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ON THE GENERAL TREATMENT OF AN OBLIQUE CRACK NEAR A BIMATERIAL INTERFACE UNDER ANTIPLANE LOADING

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Abstract—This article provides a comprehensive theoretical treatment of the interaction between an arbitrarily located and oriented crack and a bimaterial interface under antiplane loading. The analysis is based upon the use of an integral transform method and a self-consistent iterative (SCI) technique. The resulting singular integral equations are solved, using Chebyshev polynomials, to provide closed form expressions for the stress distribution along the interface and the stress intensity factor at the crack. Typical examples are provided to show the effect of the location and orientation of the crack and the material combination upon the interfacial stress distribution and the stress intensity factor of the crack. The presented method can be extended to treat more complex interaction problems. Copyright © 1996 Elsevier Science Ltd.

1. INTRODUCTION

Current development in advanced composite materials has resulted in a recent resurgence of interest in the interaction between cracks and interfaces. This interest stems mainly from the desire to characterize the modes of failure associated with cracks either lying along or terminating at a right angle to an interface.

One of the earliest articles on the interfacial crack problem is due to Williams (1959), who performed an asymptotic analysis of the elastic fields at the tip of a crack. His solution revealed the rapid oscillations in the stress and displacement fields, implying the physically impossible phenomenon of interpenetration between crack surfaces. Subsequent attempts, e.g. Sih and Rice (1964), England (1965), Rice and Sih (1965), Hutchinson *et al.* (1987), Shih and Asaro (1988), Rice (1988), Suo (1990), Bassani and Qu (1990) and Qu and Bassani (1993), have retained this oscillatory behavior and interpenetration between the crack surfaces.

In an attempt to overcome some of these difficulties, Atkinson (1977) and Delale and Erdogan (1988) developed an interface model which allows a continuously varying modulus in the interfacial layer. This model leads to the usual square root singularity and avoids the oscillatory behavior observed in earlier attempts. Comninou (1977), on the other hand, resolved the difficulties associated with the oscillatory nature of the interfacial crack problem by allowing for partial closure at the crack tips.

To obtain the stress intensity factor of a crack which is perpendicular to the interface of two bonded half-spaces and subjected to concentrated wedge loading, Cook and Erdogan (1972) used the Mellin transform method to formulate the problem and to derive the appropriate integral equations necessary to describe it. Erdogan and Biricikoglu (1973) studied the problem of two bonded half-spaces containing a finite crack perpendicular to and crossing through the interface. The problem was formulated as a system of singular integral equations with a generalized Cauchy kernel which enabled the numerical determination of the stress intensity factors. Erdogan *et al.* (1991a,b) further investigated the antiplane problem for a crack perpendicular to the interface between inhomogeneous media. Of particular relevance to the current study is the work of Bassani and Erdogan (1978) and Ashbaugh (1975). They treated the near-interface crack problem under antiplane and plane loadings, respectively, using the singular integral equation method. Due to the

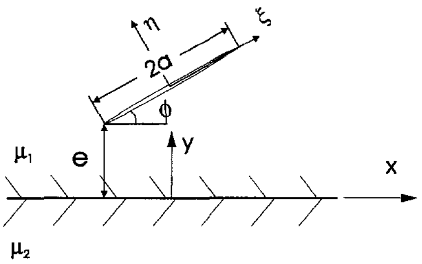


Fig. 1. Arbitrarily located and oriented crack near an interface.

difficulties associated with the formulations resulting from the complex boundary conditions, only numerical results for specialized examples were given.

The objective of this article is to present a self-consistent iterative (SCI) technique to treat the interaction between an arbitrarily located and oriented crack and an interface under antiplane loading. In this technique, which avoids dealing with the complex boundary conditions of the problem, the original problem is decomposed into a number of subproblems, each containing either the crack or the bimaterial interface. The analysis of the crack subproblem is based upon the use of Fourier transform and the solution of the resulting singular integral equations using Chebyshev polynomials, while the solution of the interface subproblem is given in a closed form using Fourier transform. The final solution is then obtained by employing a self-consistent iterative technique to the different subproblems. Two aspects of the work are accordingly examined: the first is concerned with the verification of the newly developed SCI scheme and the second with the effect of the crack location and orientation and the elastic moduli of the bimaterial upon the stress intensity factor at the crack and the stress field at the interface.

2. FORMULATION OF THE PROBLEM

Consider a crack of length $2a$ which is arbitrarily located and oriented near a perfectly bonded bimaterial interface of two dissimilar half planes. Both planes are made of linearly elastic, homogeneous and isotropic materials with the shear modulus being given by

$$\mu = \mu_1 + (\mu_2 - \mu_1)H(-y) \tag{1}$$

where $H(y)$ is a step function.

Let (x, y) and (ξ, η) be two rectangular coordinate systems with their origins at the interface and the center of the crack, respectively, as depicted in Fig. 1. The distance between the interface and the left tip of the crack is denoted e and the orientation angle of the crack ϕ.

Assuming that the bimaterial is loaded with $\tau_{xz} = \tau_1 H(y) + \tau_1(u_2/u_1)H(-y)$, $\tau_{yz} = \tau_2$ away from the crack region, the original problem can be divided into three main problems, thus avoiding the complex boundary conditions. The first one (A of Fig. 2) which involves the deformation of the bimaterial, in the absence of the crack, due to the applied stresses at infinity, has been solved. The solution leads to a general description of the shear stress $\tau^{(0)}$ along the crack surfaces such that

$$\tau^{(0)} = \tau_2 \cos\phi - \tau_1 \sin\phi. \tag{2}$$

The second problem (B of Fig. 2) pertains to a single crack in an infinite homogeneous elastic medium ($\mu = \mu_1$) and can be formulated as

$$\nabla^2 w_B^{(m)} = 0 \quad -\infty < \xi, \quad \eta < \infty \tag{3}$$

with

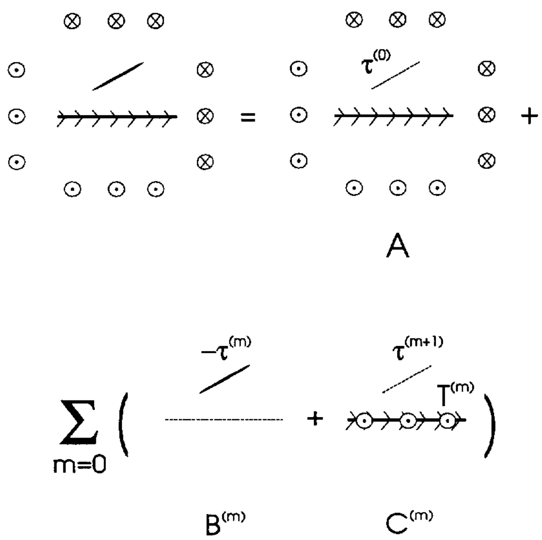


Fig. 2. Superposition of subproblems.

$$\tau_{B\eta z}^{(m)} = -\tau^{(m)} \quad |\xi| < a, \quad \eta = 0 \tag{4}$$

being the only external load applied along the crack surfaces. The third problem is concerned with the determination of the stresses in the bimaterial in the absence of the crack (C of Fig. 2), which is governed by

$$\mu \nabla^2 w_C^{(m)} + \delta(y)(\mu_1 - \mu_2) \frac{\partial w_C^{(m)}}{\partial y} + \delta(y) T^{(m)} = 0 \tag{5}$$

with

$$T^{(m)} = (\mu_1 - \mu_2) \frac{\partial w_B^{(m)}}{\partial y} \tag{6}$$

being the only external load distributed along the interface.

If the iterative procedure shown in Fig. 2 is converging, the boundary condition along the crack surfaces are satisfied since $\tau^{(\infty)} = 0$. The resulting displacement field then given by

$$w = w_A + \sum_{m=0}^{\infty} (w_B^{(m)} + w_C^{(m)}) \tag{7}$$

provides the solution of the original boundary value problem.

3. SOLUTIONS OF SUBPROBLEMS

3.1. *The single crack problem*

Let us consider the elastic behavior of a single crack in a homogeneous medium ($\mu = \mu_1$), as shown in Fig. 2. By making use of Fourier transforms, the general solution of eqn (3) can be expressed in terms of the following Fourier integrals:

$$w(\xi,\eta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} A(s)e^{-|s\eta|}e^{-is\xi}\,\mathrm{d}s \quad \eta \geqslant 0. \tag{8}$$

By introducing the following dislocation density function of the crack

$$f(\xi) = \frac{\partial w(\xi,0)}{\partial \xi} \tag{9}$$

the unknown function $A(s)$ can be found as being

$$A(s) = -\frac{\bar{f}(s)}{is} \tag{10}$$

where

$$\bar{f}(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi)e^{is\xi}\,\mathrm{d}\xi \tag{11}$$

is the Fourier transform of $f(\xi)$. Substitution of eqns (10) and (11) into eqn (8) yields

$$w(\xi,\eta) = -\frac{sgn(\eta)}{2\pi i}\int_{-\infty}^{\infty} f(u)\int_{-\infty}^{\infty}\frac{1}{s}e^{is(u-\xi)-|s\eta|}\,\mathrm{d}s\,\mathrm{d}u \tag{12}$$

and the corresponding stresses can be obtained from eqn (2) as being

$$\tau_{\eta z}(\xi,\eta) = \frac{\mu_1}{2\pi i}\int_{-\infty}^{\infty} f(u)\int_{-\infty}^{\infty} sgn(s)e^{is(u-\xi)-|s\eta|}\,\mathrm{d}s\,\mathrm{d}u \tag{13}$$

$$\tau_{\xi z}(\xi,\eta) = \frac{\mu_1 sgn(\eta)}{2\pi}\int_{-\infty}^{\infty} f(u)\int_{-\infty}^{\infty} e^{is(u-\xi)-|s\eta|}\,\mathrm{d}s\,\mathrm{d}u. \tag{14}$$

The crack is subjected to the following boundary conditions

$$w(\xi,0) = 0 \quad |\xi| \geqslant a \quad \text{and} \quad \tau_{\eta z}(\xi,0) = -\tau(\xi) \quad |\xi| < a. \tag{15}$$

By using eqns (12) and (13) in (15), the above boundary conditions can be expressed in terms of the following singular integral equations

$$\int_{-a}^{a}\frac{f(u)}{u-\xi}\,\mathrm{d}u = -\frac{\pi}{\mu_1}\tau(\xi), \quad |\xi| < a \tag{16}$$

and

$$\int_{-a}^{a} f(u)\,\mathrm{d}u = 0. \tag{17}$$

The function $f(u)$ in eqns (16) and (17) can be expressed in terms of Chebyshev Polynomials as follows:

$$f(u) = \sum_{j=0}^{\infty} \frac{c_j}{\sqrt{1-\frac{u^2}{a^2}}} T_j\left(\frac{u}{a}\right) \tag{18}$$

where T_j are Chebyshev Polynomials of the first kind and c_j are unknown constants. From the orthogonality condition of the Chebyshev Polynomials, eqn (17) results in $c_0 = 0$. Substituting (18) into (16), the following algebraic equation for c_j is obtained

$$\sum_{j=1}^{\infty} c_j U_{j-1}\left(\frac{\xi}{a}\right) = -\tau(\xi)/\mu_1, \quad |\xi| < a \tag{19}$$

where U_j represents Chebyshev Polynomial of the second kind. If the Chebyshev Polynomials in eqn (18) are truncated to the Nth term and eqn (19) is satisfied at N collocation points given by

$$\xi_l = a\cos\left(\frac{l}{N+1}\pi\right), \quad l = 1, 2, \ldots N \tag{20}$$

then eqn (19) reduces to the following linear algebraic equations

$$[\boldsymbol{\beta}]\{\mathbf{c}\} = \{\mathbf{t}\} \tag{21}$$

where the elements of $\{\mathbf{c}\}$ are the unknown Chebyshev polynomial coefficients and $[\boldsymbol{\beta}]$ and $\{\mathbf{t}\}$ are two matrices given by

$$\boldsymbol{\beta}_{lj} = \frac{\sin\dfrac{jl\pi}{N+1}}{\sin\dfrac{l\pi}{N+1}}, \quad t_l = -\tau(\xi_l)/\mu_1, \quad j, l = 1, 2, \ldots, N. \tag{22}$$

The solution of eqn (21) gives

$$\{\mathbf{c}\} = [\boldsymbol{\beta}]^{-1}\{\mathbf{t}\} \tag{23}$$

Substituting (23) and (18) into (13) and (14) and making use of the following relations ($p > 0$)

$$\int_{-1}^{1} (1-w^2)^{-1/2} T_j(w) \sin(pw)\,\mathrm{d}w = \begin{cases} 0 & j = 2n \\ (-1)^n \pi J_j(p) & j = 2n+1 \end{cases}$$

$$\int_{-1}^{1} (1-w^2)^{-1/2} T_j(w) \cos(pw)\,\mathrm{d}w = \begin{cases} 0 & j = 2n+1 \\ (-1)^n \pi J_j(p) & j = 2n \end{cases}$$

with J_j being Bessel functions of the first kind, the stress distribution resulting from the presence of the crack can be expressed as

$$\tau_{\eta z}(\xi,\eta) = \mu_1 a \sum_{j=1}^{N} c_j \begin{cases} (-1)^n \displaystyle\int_0^\infty J_j(sa)\cos(s\xi)e^{-|s\eta|}\,\mathrm{d}s & j = 2n+1 \\ (-1)^{(n+1)} \displaystyle\int_0^\infty J_j(sa)\sin(s\xi)e^{-|s\eta|}\,\mathrm{d}s & j = 2n \end{cases} \tag{24}$$

and

$$\tau_{\xi z}(\xi,\eta) = \mu_1 a\,sgn(\eta) \sum_{j=1}^{N} c_j \begin{cases} (-1)^n \displaystyle\int_0^\infty J_j(sa)\sin(s\xi)e^{-|s\eta|}\,\mathrm{d}s & j = 2n+1 \\ (-1)^n \displaystyle\int_0^\infty J_j(sa)\cos(s\xi)e^{-|s\eta|}\,\mathrm{d}s & j = 2n. \end{cases} \tag{25}$$

3.2 *The stress field due to interfacial forces*

Let us now consider the stress field introduced by a distributed force $T(x)$ along a bimaterial interface, as shown in Fig. 2. The application of the Fourier transform to eqn (6) leads to the following general solution of the present subproblem

$$w(x,y) = \begin{cases} \displaystyle\int_{-\infty}^{\infty} Ce^{-|s|y}e^{-isx}\,\mathrm{d}s & y > 0 \\ \displaystyle\int_{-\infty}^{\infty} De^{|s|y}e^{-isx}\,\mathrm{d}s & y < 0. \end{cases} \tag{26}$$

Since $w(x,0^+) = w(x,0^-)$ at $y = 0$, it follows that

$$C = D. \tag{27}$$

Then, the corresponding stresses can be expressed as

$$\tau_{yz}(x,y) = \begin{cases} -\mu_1 \displaystyle\int_{-\infty}^{\infty} |s|Ce^{-|s|y}e^{-isx}\,\mathrm{d}s & y > 0 \\ \mu_2 \displaystyle\int_{-\infty}^{\infty} |s|Ce^{|s|y}e^{-isx}\,\mathrm{d}s & y < 0 \end{cases} \tag{28}$$

$$\tau_{xz}(x,y) = \begin{cases} -i\mu_1 \displaystyle\int_{-\infty}^{\infty} sCe^{-|s|y}e^{-isx}\,\mathrm{d}s & y > 0 \\ -i\mu_2 \displaystyle\int_{-\infty}^{\infty} sCe^{|s|y}e^{-isx}\,\mathrm{d}s & y < 0. \end{cases} \tag{29}$$

The equilibrium equation implies that $T+\tau_{yz}(x,0^+) = \tau_{yz}(x,0^-)$ at $y = 0$. This leads to

$$-\mu_1|s|C+\bar{T} = \mu_2|s|C \tag{30}$$

which means that

$$C = \frac{1}{\mu_1+\mu_2}\frac{\overline{T(s)}}{|s|} \tag{31}$$

where

$$\overline{T(s)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} T(x)e^{isx}\,\mathrm{d}x \tag{32}$$

is the Fourier transform of $T(x)$. Substituting eqn (31) into (28) and (29) and making use of the following relations

$$\int_{-\infty}^{\infty} e^{-|sy|}e^{-isx}\,\mathrm{d}s = \frac{2|y|}{x^2+y^2} \tag{33}$$

and

$$\int_{-\infty}^{\infty} sgn(s)e^{-|sy|}e^{-isx}\,\mathrm{d}s = \frac{-2ix}{x^2+y^2} \tag{34}$$

the stresses caused by a distributed force T along the interface of the inhomogeneous medium can be obtained as

$$\tau_{xz}(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{x-u}{(x-u)^2+y^2}T(u)\,\mathrm{d}u\begin{cases} -\dfrac{2\mu_1}{\mu_1+\mu_2} & y>0 \\ -\dfrac{2\mu_2}{\mu_1+\mu_2} & y<0 \end{cases} \tag{35}$$

$$\tau_{yz}(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{|y|}{(x-u)^2+y^2}T(u)\,\mathrm{d}u\begin{cases} -\dfrac{2\mu_1}{\mu_1+\mu_2} & y>0 \\ \dfrac{2\mu_2}{\mu_1+\mu_2} & y<0. \end{cases} \tag{36}$$

At the interface, $y = 0$, τ_{yz} reduces to

$$\tau_{yz} = \frac{T(x)}{\mu_1+\mu_2}\begin{cases} -\mu_1 & y=0^+ \\ \mu_2 & y=0^- \end{cases} \tag{37}$$

4. A SELF-CONSISTENT ITERATIVE SOLUTION

The solution of the original boundary value problem can be obtained by the superposition of the subproblems described in the previous section. In order to obtain a self-consistent solution and avoid the complex boundary conditions of the problem, an appropriate relation between the different subproblems must be sought.

For subproblem $B^{(m)}$, the crack is subjected to the shear stress $\tau_{\eta z} = -\tau^{(m)}$ along its surface. The result given by eqn (23) provides c_j, such that:

$$\{\mathbf{c}^m\} = [\boldsymbol{\beta}]^{-1}\{\mathbf{t}^m\} \tag{38}$$

where $\{\mathbf{t}^m\} = [t_1^m, t_2^m, \ldots, t_l^m, \ldots, t_N^m]^T$, with $t_l^m = -\tau^{(m)}(\xi_l)/\mu_1$, $l = 1, 2, \ldots, N$. Accordingly, the stress distribution along the interface site can be described in terms of the undisturbed field $\tau_{\xi z}^m$ and $\tau_{\eta z}^m$ (eqns (24) and (25)) as being:

$$\tau_{yz}^m(x) = \tau_{\eta z}^m(\bar{\xi},\bar{\eta})\cos\phi + \tau_{\xi z}^m(\bar{\xi},\bar{\eta})\sin\phi = \mu_1 a \sum_{j=1}^{N} c_j^m p_j(x) \tag{39}$$

where

$$p_j(x) = \begin{cases} (-1)^n C_j(\bar{\xi},\bar{\eta})\cos\phi + \operatorname{sgn}(\bar{\eta})(-1)^n S_j(\bar{\xi},\bar{\eta})\sin\phi, & j = 2n+1 \\ (-1)^{n+1} S_j(\bar{\xi},\bar{\eta})\cos\phi + \operatorname{sgn}(\bar{\eta})(-1)^n C_j(\bar{\xi},\bar{\eta})\sin\phi, & j = 2n \end{cases} \tag{40}$$

with S_k and C_k being known functions given in the Appendix and

$$\bar{\xi} = x\cos\phi - d\sin\phi \quad \text{and} \quad \bar{\eta} = -x\sin\phi - d\cos\phi \tag{41}$$

where $d = e + a\sin\phi$.

In subproblem $C^{(m)}$, the infinite elastic medium is subjected to a distributed force along the interface resulting from subproblem $B^{(m)}$, which can be described, according to eqn (6), as

$$T^{(m)}(x) = (\mu_1 - \mu_2)a\sum_{j=1}^{N} c_j^m p_j(x). \tag{42}$$

This distributed force results in a stress field $\tau_{xz}^{(m+1)}$ and $\tau_{yz}^{(m+1)}$ which can be obtained directly from eqns (35) and (36) provided in Section 3. The stress distribution along the crack site, corresponding to this subproblem, can thus be given as

$$\tau^{(m+1)}(\xi) = \tau_{yz}^{(m+1)}(\bar{x},\bar{y})\cos\phi - \tau_{xz}^{(m+1)}(\bar{x},\bar{y})\sin\phi = \sum_{j=1}^{N} c_j^m g_j(\xi) \tag{43}$$

where

$$g_j(\xi) = \mu_1\frac{(\mu_1-\mu_2)a}{(\mu_1+\mu_2)\pi}\int_{-\infty}^{\infty}\frac{(\bar{x}-u)\sin\phi - |\bar{y}|\cos\phi}{(\bar{x}-u)^2+\bar{y}^2}p_j(u)\,\mathrm{d}u \tag{44}$$

with

$$\bar{x} = \xi\cos\phi, \quad \bar{y} = d + \xi\sin\phi. \tag{45}$$

In subproblem $B^{(m+1)}$, the crack is subjected to $\tau_{\eta z} = -\tau^{(m+1)}$ at its surfaces. The governing equation for solving c_j^{m+1} $(j = 1, 2, \ldots, N)$ can be obtained from (21) as being

$$[\boldsymbol{\beta}]\{\mathbf{c}^{m+1}\} = [\mathbf{g}]\{\mathbf{c}^m\} \tag{46}$$

where $\{\mathbf{c}^m\}$ and $\{\mathbf{c}^{m+1}\}$ are the Chebyshev polynomial coefficients of subproblems $B^{(m)}$ and $B^{(m+1)}$, and $[\mathbf{g}]$ is a matrix given by

$$g_{lj} = -g_j(\xi_l)/\mu_1, \quad j, l = 1, 2, \ldots, N \tag{47}$$

with ξ_l $(l = 1, 2, \ldots, N)$ being the collocation points given by eqn (20). The solution of eqn (46) gives

$$\{\mathbf{c}^{m+1}\} = [\boldsymbol{\alpha}]\{\mathbf{c}^m\} \tag{48}$$

where

$$[\boldsymbol{\alpha}] = [\boldsymbol{\beta}]^{-1}[\mathbf{g}] \tag{49}$$

It is interesting to note that $[\boldsymbol{\alpha}]$ is independent of the order m of the subproblems and is governed only by the geometric condition and the material properties of the original problem. This result indicates that a special relation exists between $\{\mathbf{c}^{(m+1)}\}$ and $\{\mathbf{c}^m\}$ ($m = 0, 1, 2, \ldots$) and the iteration procedure shown in Fig. 2 is self-consistent. As a result, the Chebyshev polynomial coefficients of the crack for subproblems $B^{(m)}$ can be generally expressed as

$$\{\mathbf{c}^m\} = [\boldsymbol{\alpha}]^m\{\mathbf{c}^0\} \tag{50}$$

where

$$\{\mathbf{c}^0\} = [\boldsymbol{\beta}]^{-1}\{\mathbf{t}^0\} \tag{51}$$

is the solution of problem $B^{(0)}$ with $\{\mathbf{t}^0\} = -\tau^{(0)}/\mu_1[1, 1, 1, \ldots, 1]^T$.

According to the superposition procedure given in Fig. 2, the final result of the parameters $\{\mathbf{c}\} = \{c_1, c_2 \ldots c_N\}^T$ can be obtained using the following sum

$$\{\mathbf{c}\} = \{\mathbf{c}^0\} + \{\mathbf{c}^1\} + \{\mathbf{c}^2\} + \cdots + \{\mathbf{c}^m\} + \ldots \tag{52}$$

which can be expressed in terms of $[\boldsymbol{\alpha}]$ as

$$\{\mathbf{c}\} = (\mathbf{I} + [\boldsymbol{\alpha}] + [\boldsymbol{\alpha}]^2 + \cdots + [\boldsymbol{\alpha}]^m + \ldots)\{\mathbf{c}^0\} \tag{53}$$

where $\mathbf{I}$ is the identity matrix. If the eigenvalues of the matrix $[\boldsymbol{\alpha}]$ are less than one, the sum of (53) can be rewritten as

$$\{\mathbf{c}\} = [\mathbf{q}]\{\mathbf{c}^0\} \tag{54}$$

where

$$[\mathbf{q}] = (\mathbf{I} - [\boldsymbol{\alpha}])^{-1}. \tag{55}$$

Equation (54) indicates that whilst the method treats three different subproblems, the final result of the superimposed solution is obtained directly in an analytical form in terms of the geometry and material properties considered. This avoids the commonly used alternating approach which deals with the different subproblems separately.

Accordingly, the stress intensity factor at the left tip of the crack, in the presence of the interface, can be expressed in terms of c_j ($j = 1, N$) as

$$K_{III} = \mu_1\sqrt{\pi a}\sum_{j=1}^{N}(-1)^j c_j. \tag{56}$$

The shear stress along the interface can also be obtained using the same iterative procedure. The interfacial stress due to all crack-subproblems can be expressed in the form:

$$\tau_{yz}^c(x) = \tau_{yz}^0(x) + \tau_{yz}^1(x) + \cdots + \tau_{yz}^m(x) + \ldots \tag{57}$$

By making use of eqn (39), this result can be rewritten as

$$\tau^{c}_{yz}(x) = \mu_1 a \sum_{j=1}^{N} (c_j^0 + c_j^1 + \cdots + c_j^m + \ldots) p_j(x) = \mu_1 a [(\mathbf{I} - [\boldsymbol{\alpha}])^{-1} \{\mathbf{c}^0\}]^T \{\mathbf{p}(x)\} \tag{58}$$

where $\{\mathbf{p}(x)\} = \{p_1(x), p_2(x), \ldots, p_N(x)\}^T$.

The interfacial stress due to subproblem $C^{(m)}$ can be obtained from eqn (37) as

$$\tau^{in}_{yzm}(x) = -\frac{\mu_1}{\mu_1 + \mu_2} T^{(m)}(x), \quad y = 0^+ \tag{59}$$

Referring to eqn (42), the above result can be rewritten as

$$\tau^{in}_{yzm} = -\frac{\mu_1(\mu_1 - \mu_2)a}{\mu_1 + \mu_2} \sum_{j=1}^{N} c_j^m p_j(x). \tag{60}$$

Accordingly, the interfacial stress due to all the interface-subproblems is given by

$$\tau^{in}_{yz}(x) = -\frac{\mu_1(\mu_1 - \mu_2)a}{(\mu_1 + \mu_2)} \sum_{j=1}^{N} (c_j^0 + c_j^1 + \cdots + c_j^m + \ldots) p_j(x) \tag{61}$$

$$= -\frac{\mu_1(\mu_1 - \mu_2)a}{(\mu_1 + \mu_2)} [(\mathbf{I} - [\boldsymbol{\alpha}])^{-1} \{\mathbf{c}^0\}]^T \{\mathbf{p}(x)\}.$$

Finally, the interfacial stress can be obtained by superimposing the above two results (eqns (58) and (61)), such that:

$$\tau^{IN}_{yz} = \tau^{c}_{yz} + \tau^{in}_{yz} = \frac{2\mu_1\mu_2 a}{\mu_1 + \mu_2} \{\mathbf{c}^0\}^T (\mathbf{I} - [\boldsymbol{\alpha}])^{-T} \{\mathbf{p}(x)\}. \tag{62}$$

5. RESULTS AND DISCUSSION

This section is divided into two main parts. The first deals with the verification of the resulting solutions and the second with examining the effect of the pertinent parameters upon the normalized interfacial stress ($\tau_{yz}/\bar{\tau}$) and the normalized stress intensity factor ($K^* = K_{III}/\bar{\tau}\sqrt{\pi a}$) at the left tip of the crack, with $\bar{\tau} = \tau_1 = \tau_2$ being the applied shear stress.

First, we restrict our attention to the case where the crack is perpendicular to the interface for which numerical results of the stress intensity factor of the crack have been given by Bassani and Erdogan (1979), using the singular integral equation method. The results of the normalized stress intensity factor of the crack given by Erdogan are compared with the results of the present solution in Tables 1 and 2 for $\mu_1/\mu_2 = 23.08$ and $\mu_1/\mu_2 = 0.0433$, in which $K^*(+)$ and $K^*(-)$ represent the normalized stress intensity factors

Table 1. Normalized SIF K^* for $\mu_1/\mu_2 = 0.0433$ and $\theta = 90^\circ$

	Bassani and Erdogan(1979)		Present Work	
e/a	$K^*(+)$	$K^*(-)$	$K^*(+)$	$K^*(-)$
0.00	0.907	14.0	0.907	—
0.05	0.916	0.601	0.917	0.578
0.10	0.924	0.712	0.925	0.700
0.15	0.931	0.770	0.932	0.764
0.25	0.941	0.837	0.943	0.836
0.50	0.959	0.911	0.960	0.912
1.00	0.976	0.959	0.978	0.961
4.00	0.996	0.995	0.998	0.997
9.00	0.999	0.999	0.999	0.999
∞	1.000	1.000	1.000	1.000

Table 2. Normalized SIF K^* for $\mu_1/\mu_2 = 23.08$ and $\theta = 90°$

	Bassani and Erdogan(1979)		Present Work	
e/a	$K^*(+)$	$K^*(-)$	$K^*(+)$	$K^*(-)$
0.00	1.26	0.0767	1.18	—
0.05	1.13	1.70	1.13	1.63
0.10	1.11	1.44	1.11	1.41
0.15	1.09	1.32	1.09	1.31
0.25	1.07	1.21	1.07	1.20
0.50	1.05	1.10	1.05	1.10
1.00	1.03	1.04	1.02	1.04
4.00	1.00	1.00	1.00	1.00
∞	1.00	1.00	1.00	1.00

at the right and the left tips of the crack, respectively. The comparison shows good agreement between these two solutions.

The accuracy of the solution was further verified using the limiting case of a crack parallel to the interface. When the crack tends to the interface, its strain energy release rate should be identical to that of the corresponding interfacial crack; thus, leading to the following closed form solution of the stress intensity factor

$$K_{III} = \sqrt{\frac{1}{2}\left(\frac{\mu_1}{\mu_2}+1\right)}K_{III}^{in} \tag{63}$$

where $K_{III}^{in} = \bar{\tau}\sqrt{\pi a}$ is the stress intensity factor of an interface crack of length $2a$. The normalized stress intensity factor of the near-interface crack can thus be expressed as

$$K^* = \sqrt{\frac{1}{2}\left(\frac{\mu_1}{\mu_2}+1\right)}. \tag{64}$$

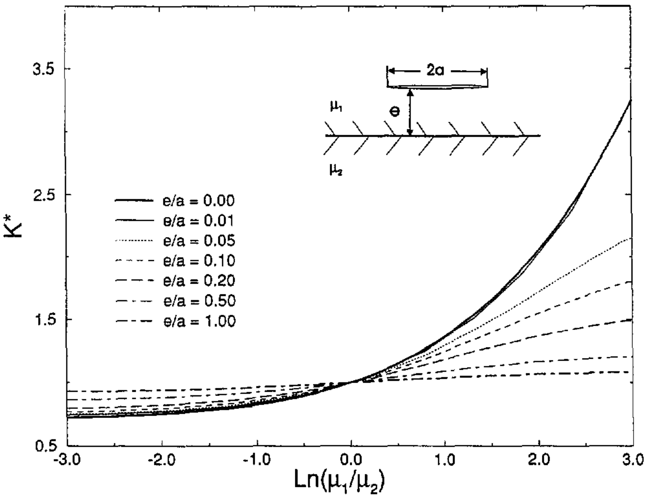
Figure 3(a) shows an excellent agreement between the closed form solution of the SIF and that resulting from SCI technique when $e/a = 0.01$.

It should be noted that the convergence condition described in (54) has been satisfied for all the examples treated in this paper. This condition is satisfied even when the distance between the interface and the crack tip is extremely small, for example when $e/a = 0.01$, as depicted in Fig. 3(a).

Consider now the stress intensity factor of an arbitrarily located and oriented crack near the interface. Figures 3(a) and (b) show the variation of the normalized stress intensity factor K^* with the shear moduli ratio μ_1/μ_2 for the case where $\phi = 0$ and $\phi = 90$, respectively. For the parallel crack case, Fig. 3(a) demonstrates the relative insensitivity of the stress intensity factor to the material combination and the position of the crack when $\mu_1 < \mu_2$. For the perpendicular crack case, Fig. 3(b) shows the strong dependence of the stress intensity factor upon the material combination and the position of the crack. The effect of the crack orientation upon K^* for different shear moduli ratios μ_1/μ_2 was also examined in Fig. 4 for a given position, $e/a = 0.1$. The figure indicates that the relationship between K^* and ϕ is linear and can be expressed as

$$K^* = k(e/a, \mu_1/\mu_2)\left(\phi - \frac{\pi}{4}\right). \tag{65}$$

In Figures 5(a) and (b), we examine the effect of the crack position e/a and the shear moduli ratio μ_1/μ_2 upon K^* for different orientation angles of the crack. Figures 5(a) and (b) show a rapid decrease in the stress intensity factor with increasing e/a for $\mu_1/\mu_2 > 1$; i.e., when the crack is in the stiffer medium. For $\mu_1/\mu_2 < 1$, K^* increases gradually with increasing e/a, though it remains below unity. The negative sign in Fig. 5(b) is due to the orientation of the crack ϕ.



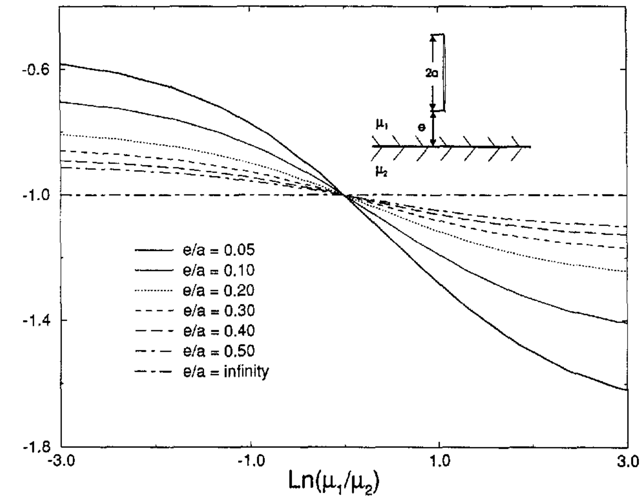


Fig. 3. Variation of normalized stress intensity factor K^* vs material combination μ_1/μ_2 for different locations of crack e/a: (a) $\phi = 0°$, and (b) $\phi = 90°$.

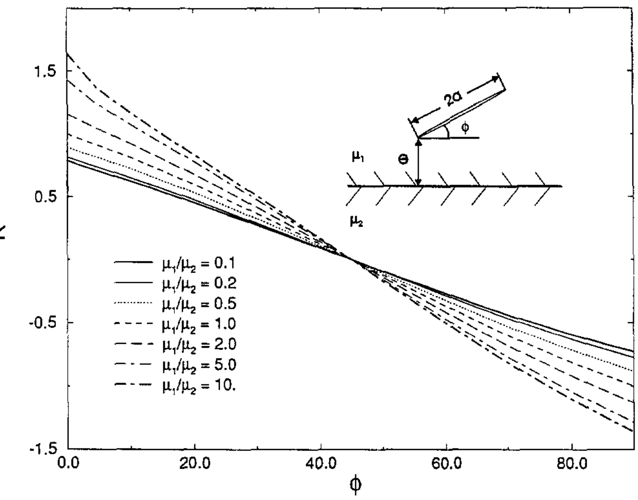


Fig. 4. Variation of normalized stress intensity factor K^* vs orientation angle ϕ for different material combinations for $e/a = 0.1$.

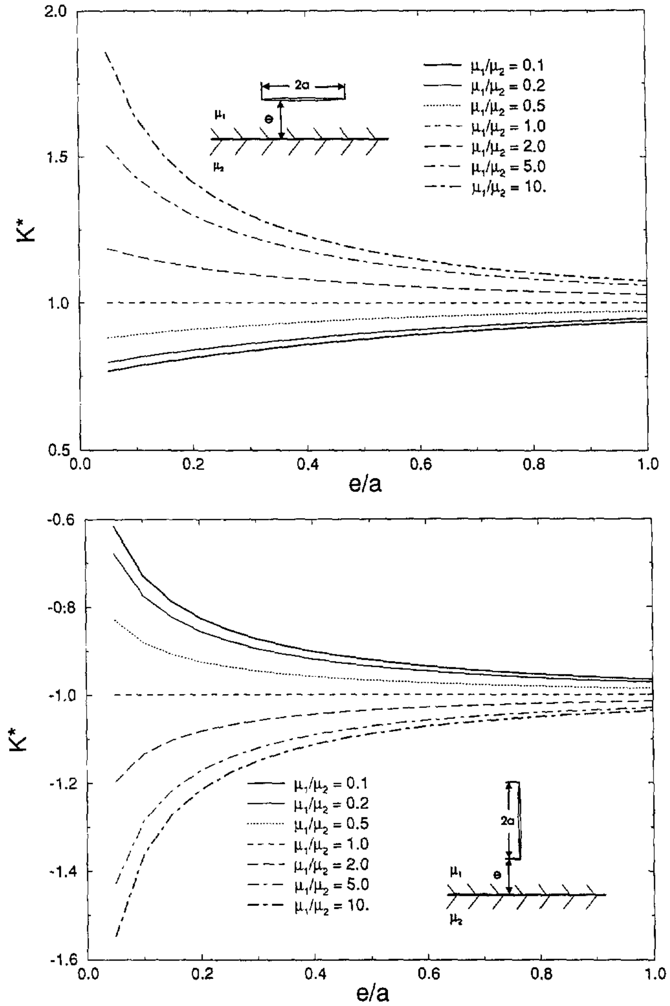


Fig. 5. Variation of normalized stress intensity factor K^* vs location of the crack e/a for different material conbinations: (a) $\phi = 0°$, and (b) $\phi = 90°$.

Let us now focus attention on the effect of the presence of a crack near the interface upon the interfacial stress. Figure 6 shows the variation of the shear stress at the interface

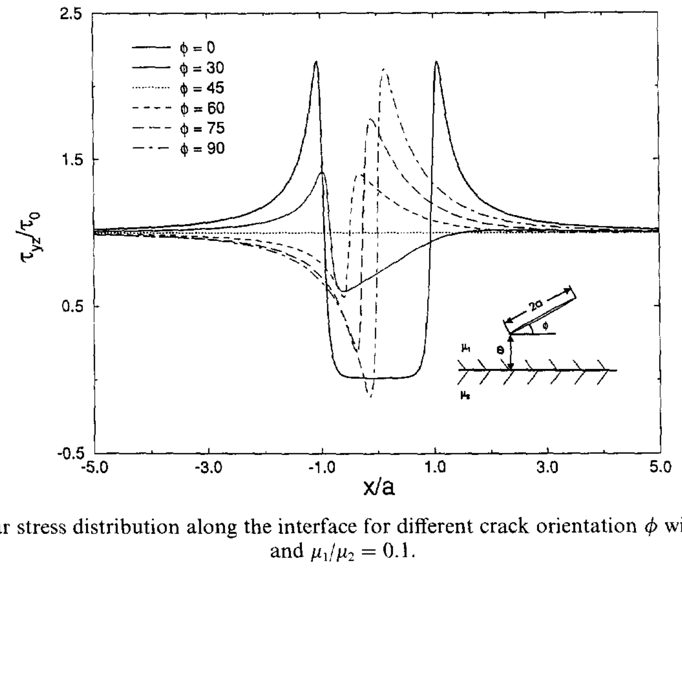


Fig. 6. Shear stress distribution along the interface for different crack orientation ϕ with $e/a = 0.1$ and $\mu_1/\mu_2 = 0.1$.

for different orientations of the crack for the case where $e/a = 0.1$ and $\mu_1/\mu_2 = 0.1$. The maximum values of the interfacial stress are observed when the crack is parallel or perpendicular to the interface. Figures 7(a) and 7(b) show the effect of the shear moduli ratio upon the interfacial stress for parallel and normal cracks, respectively. The figures show that the peak values of the interfacial stress increase with a decrease in the ratio μ_1/μ_2, i.e., the crack is near a stiffer interface.

6. CONCLUDING REMARKS

A general method is developed to describe the antiplane behavior of a crack near an interface. An analytical expression is obtained for the interfacial stress and the stress intensity factor of the crack. The analysis is based upon the use of a self-consistent iterative technique coupled with the integral transform method and the solution of singular integral equations using Chebyshev polynomials.

The complex boundary conditions of the problem are avoided by reducing the original boundary value problem into three simpler subproblems. The proposed method allows the development of a superimposed solution directly from the different subproblems. This enables the avoidance of the commonly used alternating approach which deals with the different subproblems separately. In addition, this method can be easily extended to treat different interacting problems with complex boundary conditions.

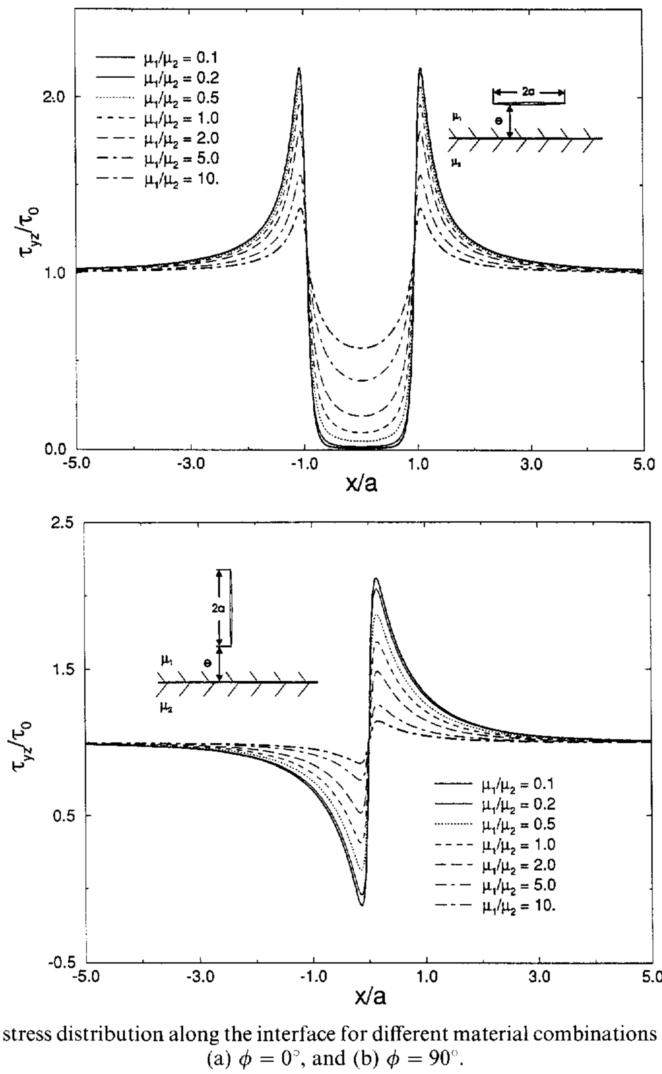


Fig. 7. Shear stress distribution along the interface for different material combinations for $e/a = 0.1$: (a) $\phi = 0^\circ$, and (b) $\phi = 90^\circ$.

The results reveal the dependence of the stress intensity factor at the crack and the shear stress at the interface upon the location and the orientation of the crack as well as the elastic mismatch of the bimaterial interface. An important finding of the work is that the stress intensity factor at the crack tip decreases, while the interfacial stress increases, with a decrease in the shear moduli ratio μ_1/μ_2.

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APPENDIX

The following expressions are used in eqn (40):

$$C_k(x,y) = \int_0^\infty J_k(as)\cos(s|x|)e^{-s|y|}\,\mathrm{d}s = \frac{a^k\cos\left(Ak-\frac{|B|}{2}\right)}{R\left[\left(R\cos\frac{|B|}{2}+|y|\right)^2+\left(R\sin\frac{|B|}{2}+|x|\right)^2\right]^{k/2}} \tag{A1}$$

$$S_k(x,y) = \int_0^\infty J_k(as)\sin(s|x|)e^{-s|y|}\,\mathrm{d}s = \frac{-a^k\sin\left(Ak-\frac{|B|}{2}\right)}{R\left[\left(R\cos\frac{|B|}{2}+|y|\right)^2+\left(R\sin\frac{|B|}{2}+|x|\right)^2\right]^{k/2}} \tag{A2}$$

where

$$R = 4\sqrt{(y^2 - x^2 + a^2)^2 + 4x^2y^2}$$

$$B = -\left|\arccos\frac{y^2 - x^2 + a^2}{R^2}\right|$$

$$A = -\operatorname{arctg}\frac{R\sin\frac{|B|}{2} + |x|}{R\cos\frac{|B|}{2} + |y|}.$$